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Nonclassical symmetry reductions of a porous medium equation with convection

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Abstract. In this paper new symmetry reductions and exact solutions are presented for the porous medium equation with convection $u_t = (u^n)_{xx} + f(x)\,u^s\,u_x$. Those spatial forms for which the equation can be reduced to an ordinary differential equation are studied. The symmetry reductions and exact solutions presented are derived by using the nonclassical method developed by Bluman and Cole and are unobtainable by Lie classical method. The asymptotic behaviour of some of the new solutions is analysed.

1. Introduction

The quasi-linear parabolic equation

$$u_t = (u^n)_{xx} + f(x)u^s u_x \tag{1}$$

with $n \neq 0$ is a mathematical model for many physical problems that corresponds to nonlinear diffusion with convection. The second term on the right-hand side of (1) is of convective nature. In the theory of unsatured porous medium, the convective part may represent the effect of gravity. When $f(x) = \text{constant}$, we arrive to the Boussinesq equation of hydrology, involved in various fields of petroleum technology and ground water hydrology.

The importance of the effect of space-dependent parts on the overall dynamics of (1) is well known. Nevertheless, more often that not, the spatial dependent factors in (1) are assumed to be constant, although there is no fundamental reason to assume so. Actually, allowing for their spatial dependence enables one to incorporate additional factors into the study which may play an important role. For instance, in porous medium this may account for intrinsic factors like medium contamination with another material or in plasma. This may express the impact that solid impurities arising from the walls have on the enhancement of the radiation channel.

It is a well known fact that non-negative solutions u of (1) may give rise to interfaces (or free boundaries) separating regions where $u > 0$ from ones where $u = 0$. These fronts are relevant in the physical problems modelled and their occurrence is essentially due to slow diffusion ($n > 1$) or to convective phenomena dominating over diffusion ($s < n - 1$). In this case if $s \leqslant 0$ and $s < n - 1$ there is a great contrast with pure diffusion phenomena [1].

For $s = 1$ we obtain a particular case of the generalized Hopf equation. Lie symmetries for this equation were obtained by Katkov [12]. When $n = 1$ in (1) we obtained the Burgers equation. Nonlocal symmetries and Lie–Bäcklund symmetries for this equation are well known [2, 11–13]. The generalized diffusion equation

$$T_t = (D_1(T)T_x)_x + a(D_2(T))_x + b(x,t)D_3(T)$$

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where $T(x,t)$ denotes the temperature at a point, a is an arbitrary constant, D_1, D_2 and D_3 are arbitrary functions of temperature T and $b(x,t)$ is another arbitrary function of x and t, has been analysed via isovector approach, and some new exact solutions have been obtained by Bhutani *et al* [4]. We recover some of the results obtained by them when $D_i(u)$, $i=1,2,3$ have a power law dependence.

Motivated by the fact that symmetry reductions for many potential differential equations (PDEs) are known that are not obtained by using the classical Lie group method, there have been several generalizations of the classical Lie group method for symmetry reductions. Bluman and Cole developed the nonclassical method to study the symmetry reductions of the heat equation; Clarkson and Mansfield [6] presented an algorithm for calculating the determining equations associated with the nonclassical method. The basic idea of the method is that the PDE (1) is augmented with the invariance surface condition

$$pu_x + qu_t - r = 0 \tag{2}$$

which is associated with the vector field

$$V = p(x,t,u)\frac{\partial}{\partial x} + q(x,t,u)\frac{\partial}{\partial t} + r(x,t,u)\frac{\partial}{\partial u}. \tag{3}$$

By requiring that both (1) and (2) are invariant under the transformation with infinitesimal generator (3) one obtains an overdetermined, nonlinear system of equations for the infinitesimals $p(x,t,u)$, $q(x,t,u)$, and $r(x,t,u)$. The number of determining equations arising in the nonclassical method is smaller than for the classical method, consequently the set of solutions is, in general, larger than for the classical method as in this method one requires only the subset of solutions of (1) and (2) to be invariant under the infinitesimal generator (3). However, the associated vector fields do not form a vector space. These methods were generalized by Olver and Rosenau [14, 15] to include ‘weak symmetries’, ‘side conditions’ or ‘differential constraints’, although their methods are too general to be practical.

In previous works, for the porous medium with convection and absorption we have derived Lie classical symmetries [7], as well as nonlocal potential symmetries [8]. We also have obtained nonclassical symmetries for the porous medium with absorption [9]. The study of another equation of type (1) appears in [10].

In this paper, which is part of a program to study the symmetries of the porous medium equation, we consider nonclassical symmetries of the porous medium equation with convection (1) by using a method due to Bluman and Cole [5].

Although most papers studying nonclassical symmetries include the classical ones, in this paper we consider nonclassical symmetries of equation (1), which are unobtainable by the Lie classical method and find conditions on $f(x)$ as well as the special values of n and s for which these reductions can be derived.

In each section we list the functions $f(x)$ and the parameters n and s for which we obtain nonclassical symmetries. We also report the reduction obtained as well as some new exact solutions.

2. Nonclassical Symmetries

To apply the nonclassical method to (1), we require (1) and (2) to be invariant under the infinitesimal generator (3). In the case $q \neq 0$, without loss of generality, we may set $q(x,t,u)=1$. The nonclassical method applied to (1) gives rise to the following determining equations for the infinitesimals

$$0 = \frac{\partial^2 p}{\partial u^2}u - n\frac{\partial p}{\partial u} + \frac{\partial p}{\partial u} \tag{4}$$

$$0 = -2f\frac{\partial p}{\partial u}u^s - n\left(\frac{\partial^2 r}{\partial u^2} - 2\frac{\partial^2 p}{\partial u \partial x}\right)u^{n-1} - (n-1)n\frac{\partial r}{\partial u}u^{n-2}$$
$$+(n-1)nru^{n-3} - 2p\frac{\partial p}{\partial u} \qquad (5)$$
$$0 = -\left(f\frac{\partial p}{\partial x} + \frac{\partial f}{\partial x}p\right)u^{s+2} - fr(s-n+1)u^{s+1} - n\left(2\frac{\partial^2 r}{\partial u \partial x} - \frac{\partial^2 p}{\partial x^2}\right)u^{n+1}$$
$$-2(n-1)n\frac{\partial r}{\partial x}u^n + \left(2\frac{\partial p}{\partial u}r - 2p\frac{\partial p}{\partial x} - \frac{\partial p}{\partial t}\right)u^2 + (n-1)pru \qquad (6)$$
$$0 = -f\frac{\partial r}{\partial x}u^{s+1} - n\frac{\partial^2 r}{\partial x^2}u^n + \left(\frac{\partial r}{\partial t} + 2\frac{\partial p}{\partial x}r\right)u - (n-1)r^2. \qquad (7)$$

Solutions of this system depend in a fundamental way on the values of n, s and on the function $f(x)$. By solving (4) we obtain

$$p = p_2(x,t)u^n - \frac{p_1(x,t)}{n}$$

and we can now distinguish several cases depending on n and s.

2.1. *Case I:* $n \neq -1, -\frac{1}{2}, \quad s \neq -1, -(n+1)$.

Solving (5), we obtain

$$r = a_1u^{s+2} + a_2u^{n+2} + a_3u^{n+1} + \frac{r_2}{u^{n-1}} + a_4u^2 + r_1u$$

where

$$a_1 = \frac{-2fp_2}{(s+1)(s+n+1)} \qquad a_2 = \frac{-2p_2^2}{(n+1)(2n+1)} \qquad a_3 = \frac{1}{n}\frac{\partial p_2}{\partial x} \qquad a_4 = \frac{2p_1p_2}{n(n+1)}.$$

Substituting p and r into (6) and (7), we obtain that p_1, p_2, r_1, r_2 and $f(x)$ are related by two conditions. The special values for which new symmetries different from Lie classical symmetries can be obtained, are depending on n and s, the following.

2.1.1. *Case I(a):* $n \neq 1, \frac{1}{2}$.

Case I(a)1: $s \neq n$. We recover classical symmetries [7].

Case I(a)2: $s = n$. From the determining equations we obtain that $r_2 = 0$ and that p_2 adopts any of the following forms: $p_2 = 0$, $p_2 = f(x)/(3n-1)$ or $p_2 = -f(x)$.

In the first two cases we recover the classical symmetries obtained in [7]; in the third case if $f = c$ where c is a constant we get $p_1 = r_1 = r_2 = 0$, $p = -cu^n$ and the solutions are defined implicitly by

$$x + ctu^n + h(u) = 0$$

with

$$uh'' + (1-n)h' = 0$$

therefore

$$u = \left(\frac{k_1 - nx}{n(ct + k_2)}\right)^{1/n}. \qquad (8)$$

2.1.2. Case I.(b): $n = 1$.

Case I(b)1: $s \neq 1$. We observe classical symmetries [7].

Case I(b)2: $s = 1$. From the determining equations we can distinguish the following subcases:

(i) If $p_2 = f/2$, it follows that $f(x) = $ constant and (1) becomes the Burgers equation. Nonclassical symmetries for this equation have been obtained by Pucci [**?**] and Arrigo *et al* [3].

(ii) If $p_2 = 0$ it follows that $r_1 = r_1(t)$ and p_1, r_1, r_2 and f are related by the following conditions

$$fr_1 + f\frac{\partial p_1}{\partial x} + \frac{\partial f}{\partial x}p_1 = 0$$

$$fr_2 - \frac{\partial^2 p_1}{\partial x^2} + 2p_1\frac{\partial p_1}{\partial x} + \frac{\partial p_1}{\partial t} = 0$$

$$-f\frac{\partial r_2}{\partial x} + \frac{\partial r_1}{\partial t} + 2\frac{\partial p_1}{\partial x}r_1 = 0$$

$$-\frac{\partial^2 r_2}{\partial x^2} + \frac{\partial r_2}{\partial t} + 2\frac{\partial p_1}{\partial x}r_2 = 0.$$

The previous equations are too difficult to be solved in general, nevertheless special solutions will be considered. Setting $r_1 = r_2 = 0$ we obtain that $p_1 = c/f$ and f must satisfy the following equation:

$$2c\frac{\partial f}{\partial x} + 2\left(\frac{\partial f}{\partial x}\right)^2 - f\frac{\partial^2 f}{\partial x^2} = 0$$

whose solution is

$$f = -\frac{c}{\sqrt{k_1}}\tanh(\sqrt{k_1}x + k_2) \qquad (k_1 > 0).$$

Consequently,

$$p = -\frac{\sqrt{k_1}}{\tanh(\sqrt{k_1}x + k_2)} \qquad (k_1 > 0)$$

and we obtain the nonclassical reduction

$$z = -\frac{\log\cosh(\sqrt{k_1}x + k_2)}{k_1} - t \qquad u = h(z) \qquad (k_1 > 0)$$

where $h(z)$ satisfies the following ordinary differential equation (ODE):

$$\frac{\partial^2 h}{\partial z^2} + ch\frac{\partial h}{\partial z} + k_1\frac{\partial h}{\partial z} = 0.$$

Its solution is

$$h(z) = \frac{1}{c}(2ck_3 + k_1^2)^{1/2}\tanh\left(\frac{1}{2}(2ck_3 + k_1^2)^{1/2}z + k_4\right) - \frac{k_1}{c}$$

which is an exact solution of (1).

(iii) If $p_2 = -f$, it follows that $p_1 = -f'/f$, $r_1 = r_2 = 0$ and f must satisfy the following equation

$$f^2 f''' - 5ff'f'' + 4(f')^3 = 0.$$

Dividing by f^3, integrating once with respect to x and making the change of variable $f = 1/g$ leads to $g'' - kg = 0$. Solving this equation we obtain:

$$f = \begin{cases} (k_2 x + k_3)^{-1} & \text{if} \quad k = 0 \\ (k_2 \sin(k_1 x) + k_3 \cos(k_1 x))^{-1} & \text{if} \quad k = -k_1^2 \\ (k_2 \sinh(k_1 x) + k_3 \cosh(k_1 x))^{-1} & \text{if} \quad k = k_1^2 \end{cases}.$$

Consequently

$$p = \begin{cases} -\dfrac{u + k_2}{k_1(x + k_2)} & \text{if} \quad k = 0 \\ \dfrac{k_1(k_3 \sin(k_1 x) - k_2 \cos(k_1 x)) - u}{k_2 \sin(k_1 x) + k_3 \cos(k_1 x)} & \text{if} \quad k = -k_1^2 \\ \dfrac{k_1(k_3 \sinh(k_1 x) - k_2 \cosh(k_1 x)) - u}{k_2 \sinh(k_1 x) + k_3 \cosh(k_1 x)} & \text{if} \quad k = k_1^2 \end{cases}.$$

(a) For $k = 0$, the family of invariant solutions is defined, implicitly, by

$$k_2 x^2 + 2k_3 x + 2(t + h)(u + k_2) = 0$$

with $h(u)$ a solution of $(u + k_2)h'' + 2h' = 0$, i.e. $h = (k_4 u + k_5)(u + k_2)^{-1}$. Therefore a nonclassical reduction is given by

$$u = -\frac{k_2 x^2 + 2k_3 x + 2k_2 t + 2k_5}{2t + 2k_4}.$$

(b) For $k < 0$, the family of invariant solutions is defined, implicitly, by

$$\frac{1}{k_1^2}(\log(-k_1 k_3 \sin(k_1 x) + k_1 k_2 \cos(k_1 x) + u)) - t - h = 0$$

with $h(u)$ a solution of

$$k_1^2 (\frac{\partial h}{\partial u})^2 + \frac{\partial^2 h}{\partial u^2} = 0$$

which is given by

$$h(u) = k_5 - \frac{1}{k_1^2} \log(k_1^2 u + k_1^2 k_4)$$

therefore a nonclassical symmetry reduction is

$$u = -\frac{k_1(k_3 \sin(k_1 x) - k_2 \cos(k_1 x) + k_1 k_4 \exp(k_1^2 t + k_1^2 k_5))}{k_1^2 \exp(k_1^2 t + k_1^2 k_5) - 1}.$$

(c) For $k > 0$ a nonclassical symmetry reduction is

$$u = -\frac{k_1(k_3 \sinh(k_1 x) - k_2 \cosh(k_1 x) + k_1 k_4 \exp(k_1^2 t + k_1^2 k_5))}{k_1^2 \exp(k_1^2 t + k_1^2 k_5) - 1}.$$

2.1.3. Case I(c): $n = \frac{1}{2}$.

Case I(c)1: $s \neq 0, -\frac{1}{2}, \frac{1}{2}$. We observe classical symmetries [7].

Case I(c)2: $s = -\frac{1}{2}$. By arranging the coefficient of the several powers of u in equations (4)–(7), it follows that $p_2 = 0$ and p_1, r_1, r_2 and f are related by the following conditions:

$$p_1 r_1 - 8 p_1 \frac{\partial p_1}{\partial x} + 2\frac{\partial p_1}{\partial t} = 0$$

$$2 p_1 r_2 - \frac{\partial r_1}{\partial x} - 2\frac{\partial^2 p_1}{\partial x^2} + 4 f \frac{\partial p_1}{\partial x} + 4\frac{\partial f}{\partial x} p_1 = 0$$

$$2\frac{\partial r_1}{\partial t} + r_1^2 - 8\frac{\partial p_1}{\partial x} r_1 = 0$$

$$2\frac{\partial r_2}{\partial t} + 2 r_1 r_2 - 8\frac{\partial p_1}{\partial x} r_2 - \frac{\partial^2 r_1}{\partial x^2} - 2 f \frac{\partial r_1}{\partial x} = 0$$

$$-\frac{\partial^2 r_2}{\partial x^2} - 2 f \frac{\partial r_2}{\partial x} + r_2^2 = 0.$$

Even though the previous equations are too complicated to be solved in general, special solutions will be considered.

(i) Choosing $p_1 = k$ and $r_1 = 0$ then $r_2 = -2f'$ and we obtain the nonclassical reduction

$$z = x + 2kt \qquad u = \left(h(z) + \frac{f(x)}{2k}\right)^2$$

where $k \neq 0$ and after integrating twice with respect to x, we have that $f(x)$ must satisfy the following Ricatti equation:

$$f' + f^2 - k_2 x - k_1 = 0 \tag{9}$$

where k_1, k_2, are constants. Now, we can distinguish the following subcases.

(a) If $k_2 = 0$, then we obtain

$$f = \begin{cases} \sqrt{k_1}\tanh(\sqrt{k_1}(x+c)) & \text{if} \quad k_1 > 0 \\ (x+c)^{-1} & \text{if} \quad k_1 = 0 \\ -\sqrt{-k_1}\tan(\sqrt{-k_1}(x+c)) & \text{if} \quad k_1 < 0 \end{cases}.$$

For any of these functions $f(x)$, we have that $h(z)$ satisfies the following ODE:

$$2kh' - 4k^2 h^2 + k_3 = 0$$

whose solutions are

$$h(z) = \begin{cases} -\dfrac{1}{2k(z+k_4)} & \text{if} \quad k_3 = 0 \\ \dfrac{1}{2k}\sqrt{k_3}\tanh(\sqrt{k_3}(z+k_4)) & \text{if} \quad k_3 \neq 0 \end{cases}$$

that lead to the exact solutions

$$u = \left(\frac{1}{2k}(\sqrt{k_1}\tanh(\sqrt{k_1}(x+c)) - \sqrt{k_3}\tanh(\sqrt{k_3}(z+k_4)))\right)^2 \tag{10}$$

if $k_1 \neq 0$ and $k_3 \neq 0$, or

$$u = \left(\frac{1}{2k}((x+c)^{-1} - \sqrt{k_3}\tanh(\sqrt{k_3}(z+k_4)))\right)^2$$

if $k_1 = 0$ and $k_3 \neq 0$.

We will consider qualitative aspects of these solutions in section 3.

(b) If $k_2 \neq 0$, we obtain that for any $f(x)$ which satisfies (9), $h(z)$ must satisfy the following Riccati equation:

$$2kh' - 4k^2h^2 + k_2 + k_4 = 0.$$

Setting $f = y_x/y$ and $h = w_z/(2\,k\,w)$ we obtain that $y(x)$ and $w(z)$ must satisfy the Airy equations $y'' - (k_2\,x + k_1)\,y = 0$ and $w'' - (k_2\,z + k_3)\,w = 0$. Hence, we obtain the nonclassical reduction

$$z = x + 2kt \qquad u = \frac{1}{4k^2}\left(\frac{w_z}{w} - \frac{y_x}{y}\right)^2$$

where the solution is given in terms of the Airy functions.

(ii) Choosing $p_1 = p_1(x)$, $r_1 = r_1(x)$ and $r_2 = r_2(x)$ we obtain

$$r = -4p'u + \left(\frac{5p_1''}{p_1} - \frac{2fp_1'}{p_1} - 2f'\right)\sqrt{u} \qquad f = \frac{3p_1'}{2p_1} - \frac{p_1''}{2p_1'} + \frac{k_0}{2p_1p_1'}$$

and p_1 must satisfy the following equation:

$$\begin{aligned}0 = &-p_1^3(p_1')^3p_1^{(5)} + (5p_1^3(p_1')^2p_1'' - 6p_1^2(p_1')^4)p_1^{(4)} + 4p_1^3(p_1')^2(p_1''')^2\\ &+(-17p_1^3p_1'(p_1'')^2 + 18p_1^2(p_1')^3p_1'' - 3p_1(p_1')^5)p_1''' + 9(p_1)^3(p_1'')^4\\ &-12p_1^2(p_1')^2(p_1'')^3 + 9p_1(p_1')^4(p_1'')^2 + 3(p_1')^6p_1''.\end{aligned}$$

We observe that this condition is satisfied if p_1 satisfies the following condition:

$$p_1' - ap_1^m - b = 0$$

with $m = 2$, $m = -2$ or $m = 0$ and we can distinguish the following subcases.

(a) If $m = 2$ and $ab \neq 0$, we obtain $p_1 = -(d/a)\tanh(d(x + c))$, where $d = \sqrt{-ab}$ and we obtain

$$\begin{aligned}p &= -\frac{d}{a}\tanh(d\,(x + c))\\ r &= -\frac{4d^2u - 8ad^2\sqrt{u}}{a\cosh^2(d(x + c))}\\ f &= d\frac{2 + \cosh(2d(x + c))}{\sinh(2d(x + c))}.\end{aligned}$$

If $h(z)$ satisfies the following ODE

$$a^2\frac{\partial^2 h}{\partial z^2} + (2d^2h - 2ad^2)\frac{\partial h}{\partial z} = 0$$

then

$$h(z) = k_3\tanh(d^2a^{-2}k_3z + k_5) + a \qquad \text{if} \quad \sqrt{k_1} \neq 0.$$

This gives us the nonclassical reduction

$$z = ad^{-2}\log\sinh(d(x + c)) + t \qquad u = (-h\sinh^{-2}(d(x + c)) - h + 2a)^2.$$

(b) If $m = 2$ and $b = 0$ then $p_1 = -a/(x + c)$, and we recover a classical symmetry [7].

(c) If $m = 0$ then $p_1 = (a + b)x + c$ and we recover a classical symmetry [7].

(d) If $m = -2$ and $ab > 0$ we can obtain p_1 implicitly from

$$d\;p_1 - a\arctan(bp_1/d) - bd(x + c) = 0$$

where $\sqrt{ab} = d$.

(e) If $m = -2$ and $ab < 0$ we can obtain p_1 implicitly from

$$2dp_1 - a\log\left(\frac{bp_1 - d}{bp_1 + d}\right) - 2bd(x + c) = 0$$

where $\sqrt{-ab} = d$.

(f) If $m = -2$ and $b = 0$ then $p_1 = d(x + c)^{1/3}$ with $d = (3a)^{1/3}$ and we obtain

$$p = -2d(x + c)^{1/3} \qquad r = (\tfrac{8}{3})d(x + c)^{-2/3}u \qquad f(x) = (\tfrac{5}{6})(x + c)^{-1}.$$

Hence a nonclassical reduction is

$$z = -\frac{3}{4d}(x + c)^{2/3} - t \qquad u = h(x + c)^{-4/3}$$

where $h(z)$ satisfies the following ODE

$$2h\frac{\partial^2 h}{\partial z^2} - \left(\frac{\partial h}{\partial z}\right)^2 + 16d^2h^{3/2}\frac{\partial h}{\partial z} = 0$$

whose solution is

$$h(z) = \frac{k_1}{8d^2}\tanh^2\left(\sqrt{2k_1}d(z + k_2)\right)$$

with $k_1 > 0$, that leads to the exact solutions

$$u = \frac{k_1}{8d^2}\tanh^2\left(\sqrt{2k_1}d(z + k_2)\right)(x + c)^{-4/3}.$$

(iii) Choosing $p_2 = r_2 = 0$ and $p_1 = p_1(x)$, we obtain $r_1 = -4p_1'$ and $f = (5p_1' - 2k_1)/(2p_1)$, where p_1 satisfies

$$p_1^2p_1'' + 2p_1(p_1')^2 - 2k_1p_1p_1' - k_2p_1 = 0$$

which leads to a classical reduction [7].

Case I(c)3: $s = 0$. From the determining equations we obtain that $p_2 = 0$ and p_1, r_1, r_2 and f are related by the following conditions:

$$2p_1r_1 - fr_1 - 16p_1\frac{\partial p_1}{\partial x} + 4f\frac{\partial p_1}{\partial x} + 4\frac{\partial p_1}{\partial t} + 4\frac{\partial f}{\partial x}p_1 = 0$$

$$2p_1r_2 - fr_2 - \frac{\partial r_1}{\partial x} - 2\frac{\partial^2 p_1}{\partial x^2} = 0$$

$$-2f\frac{\partial r_1}{\partial x} + 2\frac{\partial r_1}{\partial t} + r_1^2 - 84\frac{\partial p_1}{\partial x}r_1 = 0$$

$$-2f\frac{\partial r_2}{\partial x} + 2\frac{\partial r_2}{\partial t} + 2r_1r_2 - 8\frac{\partial p_1}{\partial x}r_2 - \frac{\partial^2 r_1}{\partial x^2} = 0$$

$$r_2^2 - \frac{\partial^2 r_2}{\partial x^2} = 0.$$

Setting $r_1 = 0$ and $p_1 = p_1(x)$ we obtain $p_1 = ax + b$ and we obtain

$$p = -2(ax + b) \qquad r = 6a^2\sqrt{u}(ax + b)^{-2} \qquad f = 2(ax + b).$$

Therefore we obtain the nonclassical reduction

$$z = -\frac{1}{2a}\log(ax + b) - t \qquad u = \left(\frac{3a}{4(ax + b)^2} + h\right)^2$$

where $h(z)$ satisfies the ODE

$$h'' + 2ah' + 24a^2h = 0$$

whose solution is

$$h(z) = k_1\exp(4az) + k_2\exp(-6az)$$

which leads to the exact solution

$$u = \tfrac{1}{16}(ax + b)^{-4}\exp(-8at)(4k_2\exp(10at)(ax + b)^5 + 3a\exp(4at) + 4k_1)^2.$$

Case I.(c).4: $s = \frac{1}{2}$. The reduction obtained is a particular case of the one obtained in case I(a)2.

2.2. Case II: $n = -1$.

2.2.1. Case II(a): $s \neq -1$. For $f(x) = (ax+b)^m$ we recover the classical symmetries [7].

2.2.2. Case II(b): $s = -1$. For $f = c$, from the determining equations we obtain the nonclassical symmetries $r = 0$, $p = -c/u$. Then the solutions are defined implicitly by $xu + ct - h(u) = 0$ where $h(u)$ is a solution of $h'' = 0$ and therefore the solution it is given by

$$u = \frac{ct - k_3}{x - k_1}.$$

It must be observed that this solution can in fact be obtained via classical symmetry reductions.

2.3. Case III: $n = -\frac{1}{2}$.

2.3.1. Case III(a): $s \neq -\frac{1}{2}$. For $f(x) = (ax+b)^m$ we recover the classical symmetries [7].

2.3.2. Case III(b): $s = -\frac{1}{2}$. For $f = c$, from the determining equations we obtain that $r = 0$, $p = -c/\sqrt{u}$. Then the solutions are defined implicitly by $x\sqrt{u} + ct - h(u) = 0$, with $h(u)$ a solution of $2uh'' + h' = 0$ and therefore it is given by

$$u = \left(\frac{ct + k_1}{x - k_2}\right)^2.$$

The nonclassical reduction adopts the form (8) with $n = \frac{1}{2}$.

3. Qualitative analysis of some solutions

In this section we will consider several qualitative aspects of some of the solutions we have found.

In case I(c)2(ii) equation (1) becomes

$$u_t = (u^{1/2})_{xx} + 2\sqrt{k_1}\tanh(\sqrt{k_1}(x+c))(u^{1/2})_x. \tag{11}$$

These cases corresponds to fast diffusion and nonlinear and nonhomogenous convection.

The stationary solutions of equations (11) are solutions of the ODE

$$(\hat{u}^{1/2})_{xx} + 2\sqrt{k_1}\tanh(\sqrt{k_1}(x+c))(\hat{u}^{1/2})_x = 0.$$

If we set $v = (\hat{u}^{1/2})_x$ then v must satisfy

$$v_x + 2\sqrt{k_1}\tanh(\sqrt{k_1}(x+c))v = 0.$$

The solutions of this equation are of the form

$$v = (\hat{u}^{1/2})_x = \frac{D}{\cosh^2(\sqrt{k1}(x+c))}$$

where D is an arbitrary constant. Therefore,

$$\hat{u}^{1/2} = \frac{D}{\sqrt{k_1}}\tanh(\sqrt{k_1}(x+c)) + E$$

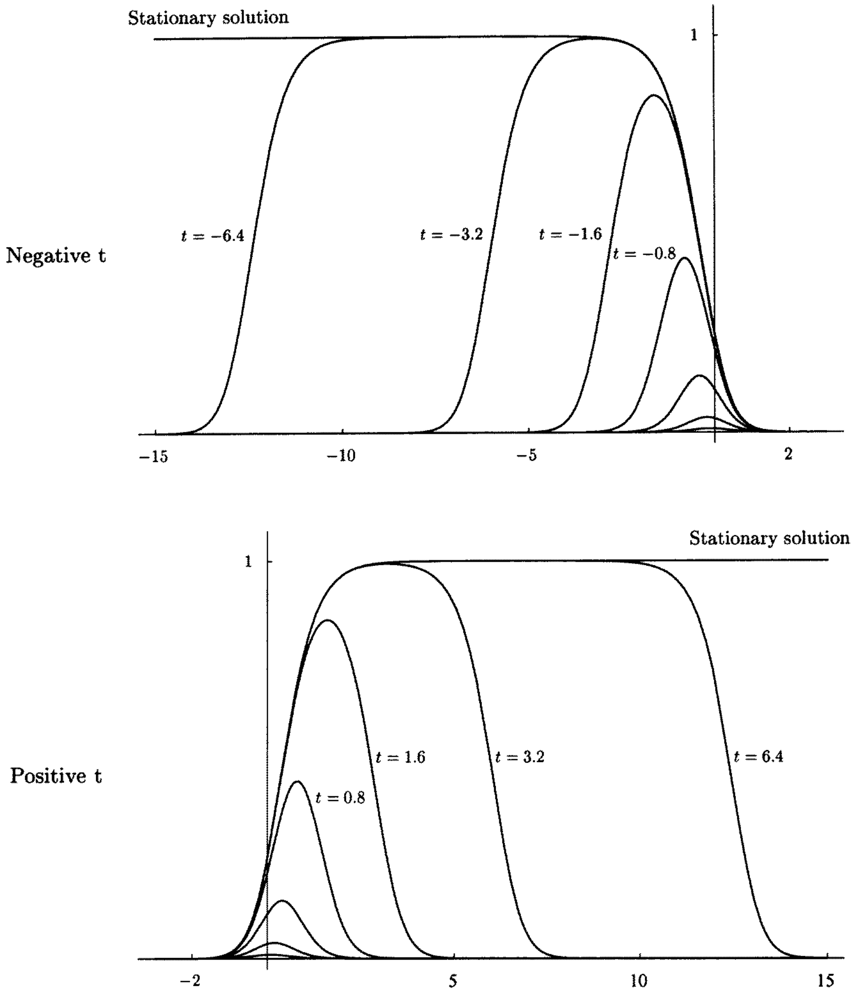

Figure 1. Solutions of (10) with $k_1 = k_3 = 1$, for several values of t and stationary solutions.

where E is also an arbitrary constant. Consequently, the stationary solutions of (11) are of the form

$$\hat{u} = \left(\frac{D}{\sqrt{k_1}} \tanh(\sqrt{k_1}(x + c)) + E \right)^{1/2}. \tag{12}$$

Let us compare these stationary solutions with the solutions $u(x, t)$ of (11) given by (10). If we set $D = k_1/k$ and $E = -\sqrt{k_3}/k$, then we observe that $\lim_{t\to\infty} u(x, t) = \hat{u}(x)$ for every $x \in R$. Therefore, the class of solutions (10) is a special class of solutions that continuously evolves towards the bi-parametric class of stationary solutions (12).

In figures 1–3 we represent the values of the solutions of (11) for several values of t altogether with the corresponding stationary solution. We easily observe that the family of solutions (10) describes processes of annihilation and subsequent creation of exponentially localized structures which asymptotically approach to a nonlocalized structure.

The class of solutions of equation (9) that is obtained for $k_1 > 0$ is qualitatively different to the one obtained for $k_1 = 0$. In this last case we obtain a different class of equations of

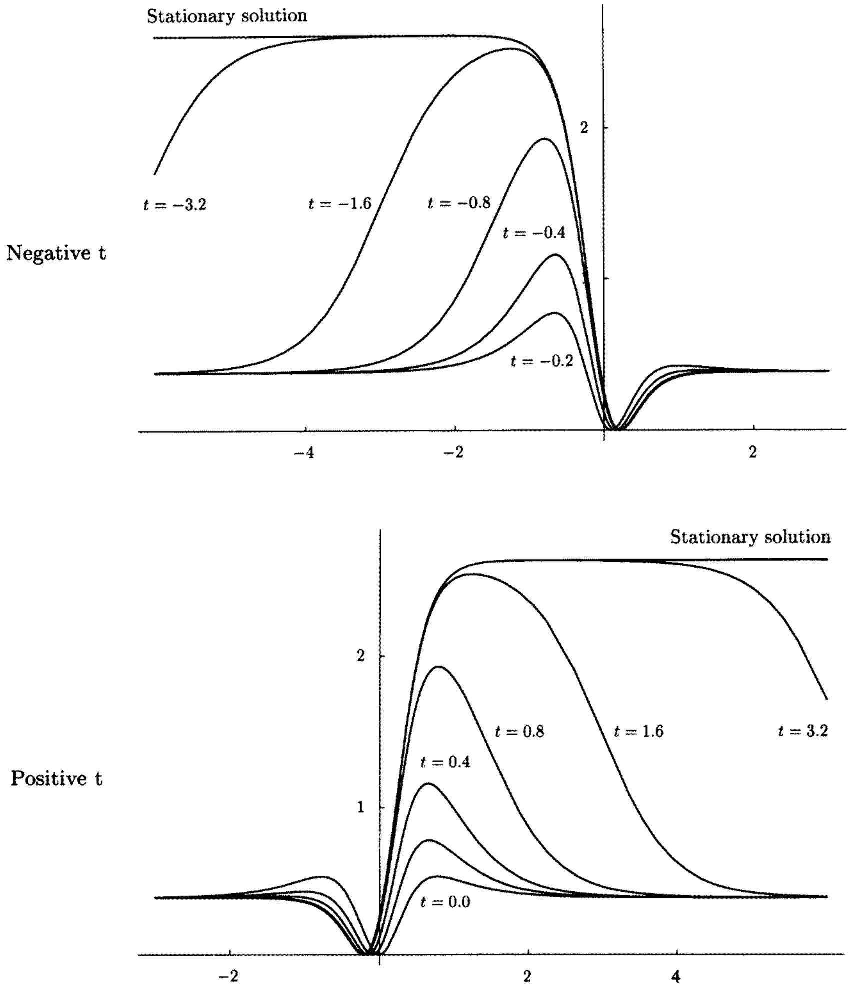

Figure 2. Solutions of (10) with $k_1 = 5$ and $k_3 = 1$, for several values of t and stationary solutions.

type (1), for which nonclassical symmetries appear.

Nevertheless, we can consider k_1 as a parameter in equation (11) and in the solutions (10). Let us observe that the second term in the right-hand side of this equation tends to zero as $k_1 \to 0$. The limit equation is the classic porous medium equation

$$u_t = (u^{1/2})_{xx}. \tag{13}$$

If in the right-hand side of (10) we let $k_1 \to 0$ then we obtain

$$u(x,t) = \left(-\frac{\sqrt{k_3}}{2k}\tanh(\sqrt{k_3}(z + k_4))\right)^2.$$

It is straightforward to check that this class of functions are solution of (13). This type of solutions corresponds to wavefronts and apparently does not appear in the literature.

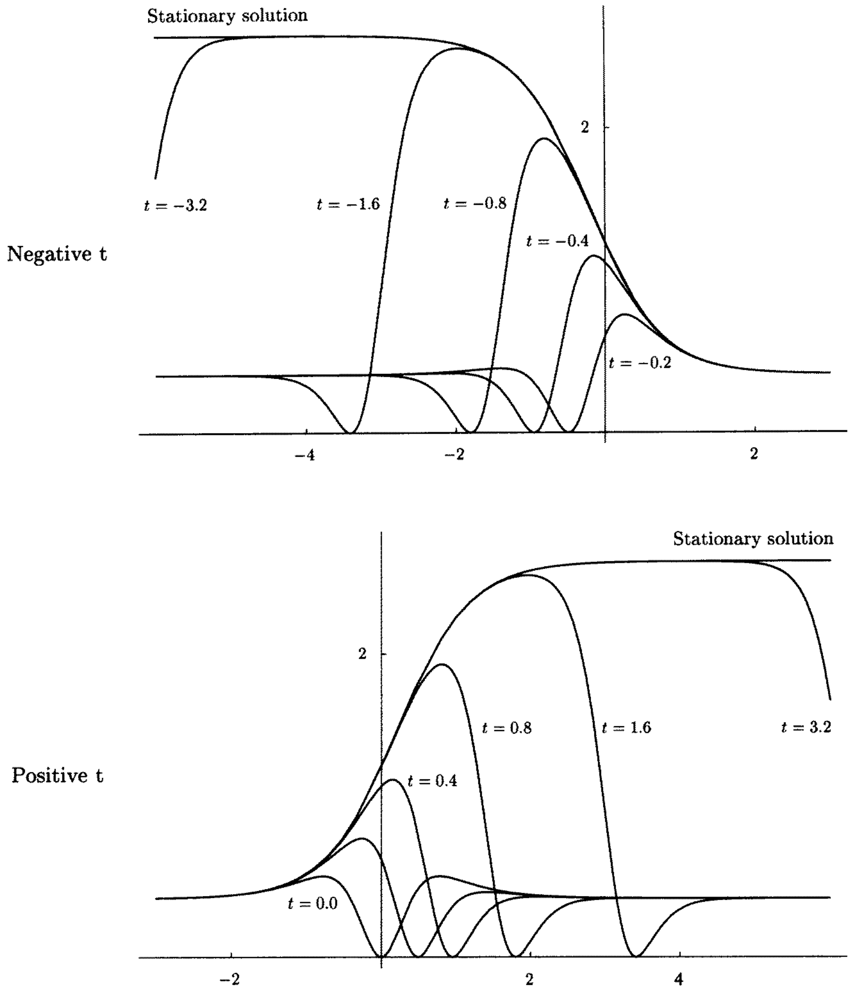

Figure 3. Solutions of (10) with $k_1 = 1$ and $k_3 = 5$, for several values of t and stationary solutions.

4. Concluding remarks

In this paper, which is part of a program to study the symmetries of the porous medium equation, we have derived the nonclassical symmetries of the quasi-linear parabolic equation (1) by using a method due to Bluman and Cole [5]. In previous works, for the porous medium with convection and absorption we have derived Lie classical symmetries [7], as well as nonlocal potential symmetries [8]. We also have obtained nonclassical symmetries for the porous medium with absorption [9]. Recognizing the importance of the space-dependent parts on the overall dynamics of (1), we have studied those spatial forms as well as the different choices for the constants n and s for which equation (1) admits the nonclassical symmetry group. We have then constructed new invariant solutions, as well as new ODEs to which (1) is reduced. These new solutions are unobtainable by the method of Lie classical symmetries. In a forthcoming paper we will deduce the symmetries of the porous medium equation in an inhomogeneus medium, to see if the known symmetries are preserved as well as if new symmetries arise.

The stationary solutions of equation (11) have been obtained as limit of the class (10) of solutions of (11). A new class of solutions of the classical porous medium equation (13) is also obtained as limit of some of our solutions.

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